MATH3705 Tutorial 6

1. Consider the Sturm-Liouville problem $y'' + 4y' + 2\lambda y = 0, 2y(0) + y'(0) = 0, y(2) = 0$.

   (a) Find all eigenvalues λ_n and corresponding eigenfunctions $y_n(x)$.

   (b) Place the equation in the Sturm-Liouville form $(py')' - qy + \lambda ry = 0$, and determine the weight function.

   (c) Find the coefficients in the expansion $f(x) = e^{-2x} = \sum_{n=1}^{\infty} c_n y_n(x), 0 < x < 2$, where $y_n(x)$ are the eigenfunctions found in part (a) with the arbitrary constants set equal to 1.

   Solution: (a) The indicial (characteristic) eqn is: $r^2+4r+2\lambda = 0$, $r = -2\pm\sqrt{2(2-\lambda)}$.

   (i) $\lambda = 2$. Then $r = -2$. $Y(x) = (A + Bx)e^{-2x}$. From the boundary conditions, $A = 0$ and $B = 0$. Thus, $\lambda = 0$ is not an eigenvalue.

   (ii) $\lambda < 2$. Let $\lambda = 2 - \omega^2$, $r = -2 \pm \sqrt{2}\omega$, $y(x) = Ae^{(-2+\sqrt{2}\omega)x} + Be^{(-2-\sqrt{2}\omega)x}$. From the boundary conditions, $A = 0$ and $B = 0$. Thus, $\lambda < 2$ is not an eigenvalue.

   (iii) $\lambda > 2$. Now let $\lambda = 2+\omega^2$, $r = -2\pm i\sqrt{2}\omega$, $y(x) = e^{-2x}(A\cos\sqrt{2}\omega x + B\sin\sqrt{2}\omega x)$. We have

   $$y(0) = A, y'(0) = -2A + \sqrt{2}\omega B.$$

   By $2y(0)+y'(0) = 0$ we imply that $B = 0$ and $y(x) = e^{-2x}A\cos\sqrt{2}\omega x$. Then by $y(2) = 0$ we have

   $$\cos 2\sqrt{2}\omega = 0 \Rightarrow \omega = \frac{(2n+1)\pi}{2\sqrt{2}}, \Rightarrow \lambda_n = 2 + \frac{(2n+1)^2\pi^2}{32}.$$

   $$y_n(x) = Ae^{-2x}\cos\left(\sqrt{2}\frac{(2n+1)\pi}{2\sqrt{2}}x\right) = Ae^{-2x}\cos\left(\frac{(2n+1)\pi x}{2}\right).$$

   (b) $r(x) = e^{4x}$.

   (c) $c_n = \frac{\int_0^2 f(x)y_n(x)r(x)dx}{\int_0^2 y_n^2(x)r(x)dx} = \frac{4(-1)^n}{(2n+1)\pi}$.

2. Integrate $\int_1^2 x^8 J_7(x)dx$.

Solution:
$$\int_1^2 x^8 J_7(x)dx = x^8 J_8(x)|_1^2 = 256J_8(2) - J_8(1).$$

3. Represent $J_3(x)$ by using $J_0(x)$ and $J_1(x)$.

Solution: From
$$J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x} J_\nu(x)$$
we imply that
$$\begin{aligned} J_0(x) + J_2(x) &= \frac{2}{x} J_1(x), \\ J_1(x) + J_3(x) &= \frac{4}{x} J_2(x). \end{aligned}$$
Hence
$$\begin{aligned} J_2(x) &= \frac{2}{x} J_1(x) - J_0(x), \\ J_3(x) &= \left(\frac{8}{x^2} - 1\right) J_1(x) - \frac{4}{x} J_0(x). \end{aligned}$$